Journal of Pure and Applied Algebra 43 (1986) 173–178
North-Holland

QUADRATIC SPACES OVER DISCRETE HODGE ALGEBRAS

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Communicated by C. Weibel
Received 21 February 1985

Introduction

Ton Vorst [9] proved the analogue of Serre's conjecture for discrete Hodge algebras over fields (R is called a *discrete Hodge algebra* over A if $R=A[X_0,\dots,X_n]/I$ where I is an ideal generated by monomials). In [1] some more results about modules over polynomial rings were extended to discrete Hodge algebras.

Here we shall extend similar results about quadratic spaces to discrete Hodge algebras. We shall prove two theorems (Theorem 2.2 and 2.4), both are extensions of Parimala's theorems (see [3,4]). In Theorem 2.2 we prove that *all quadratic spaces over discrete Hodge algebras over dedekind domains, which contain an unimodular isotropy element, are extendable from the base ring.* Theorem 2.4 states that *quadratic spaces over discrete Hodge algebras with sufficient Witt index are cancellative.*

1. Preliminaries

In this section we shall record some preliminaries. For other prerequisites and notations a reader is referred to [1] and [4].

Throughout this paper, all rings considered are commutative notherian in which 2 is invertible. A and R will always denote a ring of this kind. Δ_n and Σ will respectively denote the standard n-simplex $\{0,1,\dots,n\}$ and a simplicial subcomplex of Δ_n.

A *quadratic space* (Q,q) over A is a finitely generated projective A-module Q together with a map $q:Q\to A$ with the following properties:

(i) $q(ax)=a^2q(x)$ for all a in A and x in Q.

(ii) If $B(x,y)=q(x+y)-q(x)-q(y)$, then B is a bilinear form and B induces an isomorphism $Q\to Q^*$.

Often we suppress Q and denote (Q,q) by q.

Definition 1.1. An A-algebra R is said to be a *discrete Hodge algebra* over A if R is isomorphic to $A[X_0, X_1, \dots, X_n]/I$ where I is an ideal of $A[X_0, X_1, \dots, X_n]$ generated by monomials.

Example 1.2. Given a simplicial subcomplex Σ of Δ_n and a ring A, we shall construct a discrete Hodge algebra $A(\Sigma)$ in the following way.

Let $I(\Sigma)$ be the ideal of $A[X_0, \dots, X_n]$ generated by all square free monomials $X_{i_1}X_{i_2}\cdots X_{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r \le n$ and $\{i_1, \dots, i_r\}$ is not a face of Σ. By $A(\Sigma)$ we denote the discrete Hodge algebra $A[X_0, \dots, X_n]/I(\Sigma)$.

Various properties of $I(\Sigma)$ and $A(\Sigma)$ were discussed in, for example, [6, 8, 9, 1]. We shall quote some of them. The following easy proposition can be found in [6].

Proposition 1.3. *Let A be a commutative ring and Σ, Σ' subcomplexes of Δ_n.*

(i) *The correspondence Σ to $I(\Sigma)$ is a one-to-one correspondence between all simplicial subcomplexes of Δ_n and all ideals of $A[X_0, \dots, X_n]$ which are generated by square-free monomials.*

(ii) *The notation $A(\Sigma)$ is well defined with respect to the embedding $\Sigma \subseteq \Delta_n \subseteq \Delta_{n+1}$. In fact, $(I(\Sigma), X_{n+1})$ corresponds to Σ when it is considered as a subcomplex of Δ_{n+1}.*

(iii) *If Σ' is a subcomplex of Σ, then $I(\Sigma') \supseteq I(\Sigma)$ and hence there is a natural surjection $A(\Sigma) \to A(\Sigma')$.*

(iv) *$I(\Sigma \cup \Sigma') = I(\Sigma) \cap I(\Sigma')$ and $I(\Sigma \cap \Sigma') = I(\Sigma) + I(\Sigma')$.*

(v) *If $C(\Sigma)$ denotes the cone over Σ with vertex $n+1$, then $A(C(\Sigma)) = A(\Sigma)[X_{n+1}]$.*

The next lemma is due to Vorst [9, 3.4 Lemma, 8, Appendix].

Lemma 1.4. *Let Σ be a simplicial subcomplex of Δ_n and A be a ring. Suppose $\Sigma_1 = \Sigma \cap \Delta_{n-1}$. Then there exist subcomplexes $\Sigma_0 \subseteq \Sigma_1$ and $\Sigma_2 \subseteq \Sigma$ with*

(i) *Σ_2 is the cone over Σ_0 with n as the vertex;*

(ii) *$I(\Sigma) = I(\Sigma_1) \cap I(\Sigma_2)$ and $I(\Sigma_0) = I(\Sigma_1) + I(\Sigma_2)$.*

Also diagram

$$\begin{tikzcd} A(\Sigma) \arrow[r, "i_1"] \arrow[d, "i_2"'] & A(\Sigma_1) \arrow[d, "j_1"] \\ A(\Sigma_2) = A(\Sigma_0)[X_n] \arrow[r, "j_2"] & A(\Sigma_0) \end{tikzcd}$$

is a fibre product diagram of rings. Here all the maps are the natural surjections and j_2 is the retraction sending X_n to zero.

Lemma 1.5. *With the notations as in Lemma 1.4, if q is a quadratic space over $A(\Sigma)$, then diagram*

$$\begin{tikzcd} q \arrow[r] \arrow[d] & q\otimes A(\Sigma_1) \arrow[d] \\ q\otimes A(\Sigma_2) \arrow[r] & q\otimes A(\Sigma_0) \end{tikzcd}$$

is a fibre product diagram.

Proof. It is immediate consequence of Lemma 1.4 and the proof of [2, §2, Theorem 2.2]. □

2. The main theorems

Definition 2.1. Let (Q, q) be a quadratic space and x be an element in Q. We call x an *isotropic* element if $q(x)=0$. If further x is unimodular, then we call x a *unimodular isotropy* element.

Theorem 2.2. *Let R be a discrete Hodge algebra over a Dedekind domain A. Then if (Q, q) is a quadratic space over R and Q contains a unimodular isotropy element, then (Q, q) is extended from A.*

Proof. Let $R=A[X_0,\dots,X_n]/I$. Now $\sqrt{I}$ is generated by square-free monomials and hence $\sqrt{I}=I(\Sigma)$ for some simplicial subcomplex Σ of Δ_n (see Proposition 1.3). Let $\bar{q}=q/(X_0,\dots,X_n)q$. We want to prove $q \xrightarrow{\sim} \bar{q}\otimes_A R$. But any isometry $q/\sqrt{I}q \to \bar{q}\otimes_A R/\sqrt{I}R$ can be lifted to an isometry of q to $\bar{q}\otimes R$. Hence we can assume $R=A[X_0,\dots,X_n]/I(\Sigma)=A(\Sigma)$ for some simplicial subcomplex Σ of Δ_n. Since $\dim A(\Sigma)=\dim A$ would imply $A(\Sigma)=A$, we shall also assume that $\dim R>\dim A$.

By induction on n we shall prove that for any simplicial subcomplex Σ of Δ_n and $r\geq 0$, if q is a quadratic space over $R=A(\Sigma)[T_1,\dots,T_r]$ which contains a unimodular isotropy element, then q is extended from A.

If $n=0$, R is a polynomial ring and the statement is the theorem of Parimala [3, Theorem 3.2].

Assume $n>0$. Let $\Sigma_1, \Sigma_2, \Sigma_0$ be as in Lemma 1.4. Let $R_1=A(\Sigma_1)[T_1,\dots,T_r]$, $R_2=A(\Sigma_2)[T_1,\dots,T_r]=A(\Sigma_0)[X_n,T_1,\dots,T_r]$ and $R_0=A(\Sigma_0)[T_1,\dots,T_r]$. The diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_1 \arrow[d] \\ R_2 \arrow[r] & R_0 \end{tikzcd}$$

is a fibre product diagram of rings. And if $q_1 = q \otimes R_1$, $q_2 = q \otimes R_2$, $q_0 = q \otimes R_0$, then by Lemma 1.5 the diagram

$$\begin{tikzcd} q \arrow[r] \arrow[d] & q_1 \arrow[d] \\ q_2 \arrow[r] & q_0 \end{tikzcd}$$

is a fibre product diagram of quadratic spaces.

By induction hypothesis, q_1 is extended from A. Hence there is an isometry $f_1 : q_1 \to \bar{q} \otimes_A R_1$. Then $f_0 = f_1 \otimes_{R_1} \mathrm{Id}_{R_0} : q_0 \to \bar{q} \otimes R_0$ is also an isometry. Again as $A(\Sigma_2) = A(\Sigma_0)[X_n]$ and $\Sigma_0 \subsetneq \Delta_{n-1}$, by induction q_2 is extended from A and hence q_2 is extended from R_0. Since $\bar{q} \otimes R_2$ is also extended from R_0, there is an isometry $f_2 : q_2 \to \bar{q} \otimes R_2$ which is a lift of f_0.

Now we can construct the following fibre product diagram:

$$\begin{tikzcd} q \arrow[rr] \arrow[dd] \arrow[dr, "f"] & & q_1 \arrow[dd] \arrow[dr, "f_1"] & \\ & \bar{q} \otimes_A R \arrow[rr] \arrow[dd] & & \bar{q} \otimes_A R_1 \arrow[dd] \\ q_2 \arrow[rr] \arrow[dr, "f_2"] & & q_0 \arrow[dr, "f_0"] & \\ & \bar{q} \otimes_A R_2 \arrow[rr] & & \bar{q} \otimes_A R_0 \end{tikzcd}$$

The map $f : q \to \bar{q} \otimes_A R$ is found by the properties of fibre product. Again as f_1 and f_2 are isometries, f is also an isometry. This completes the proof of the theorem. $\square$

Remark 2.3. The proof shows that whenever an extendibility theorem for quadratic spaces over polynomial rings is available, one can extend it to discrete Hodge algebras. For example, the extendibility theorem of Rao [5] can also be extended.

The next Theorem 2.4 is about cancellation of quadratic spaces over discrete Hodge algebras. The theorem is an improvement of the cancellation theorem of Roy [7], where the Witt index of the quadratic space is assumed to be strictly greater than the dimension of the ring. In the polynomial case this theorem is due to Parimala [4] and our proof is also along the line of Parimala's proof. We omit the standard definitions and notations like generalized dimension function, isotopy between isometries, Witt index, $EO(q, H(P))$ etc. All of these can be found, for example, in [4].

Theorem 2.4. *Let R be a discrete Hodge algebra over a noetherian commutative ring A with $\dim R > \dim A$. Then any quadratic space q over R with Witt index $\geq \dim R$ is cancellative.*

Proof. Since the isometries modulo a nilpotent ideal can be lifted, we assume that A is reduced. Again by the same argument, if $R=A[X_0,\dots,X_n]/I$, replacing I by $\sqrt{I}$, we can assume that I is generated by square free monomials and hence $I=I(\Sigma)$ and $R=A(\Sigma)$ where Σ is a simplicial subcomplex of Δ_n (see Proposition 1.3).

It is enough to prove that for a quadratic form q' over $R=A(\Sigma)$ and u an unit in A, $q'\perp\langle u\rangle \xrightarrow{\sim} q\perp\langle u\rangle$ implies $q'\xrightarrow{\sim} q$, where $\langle u\rangle$ denotes the quadratic space defined by the map $R\to R$ sending x to ux^2. As $\dim A(\Sigma)>\dim A$, Σ is non empty [1, Proposition 1.5] and hence $R=A(\Sigma)$ contains a nonzero divisor t in $(X_0,\dots,X_n)R$ [1, Proposition 1.6].

Write $(Q,q)=q_0\perp H(P)$ where P is a projective R-module of rank $\geq \dim R$ and $H(P)$ denotes the hyperbolic space of P. Let $\varphi: q'\perp\langle u\rangle\to q\perp\langle u\rangle$ be an isometry and $\varphi(0,1)=(z,\theta)$. As rank $P/tP>\dim R/tR$, by [4, Corollary 3.2] there exist η_0 in $E0(q_0/tq_0\perp\langle u\rangle, H(P/tP))$ such that $\eta_0(\tilde{z},\tilde{\theta})=(0,1)$ (where $\tilde{}$ always means 'modulo t'). As η_0 can be lifted to an η in $E0(q_0\perp\langle u\rangle, H(P))$, by replacing φ by $\eta\varphi$ we can assume $\varphi(0,1)\equiv(0,1)$ 'modulo t'. Hence φ induces an isometry $q'/tq'\to q/tq$ and hence an isometry $\bar{\varphi}:\bar{q}'\to\bar{q}$ ('$\bar{}$' always means 'modulo $(X_0,\dots,X_n)$').

Write $S=$ the set of all nonzero divisors of A. As A is a reduced ring, $S^{-1}R=S^{-1}A(\Sigma)\approx(k_1\times k_2\times\cdots\times k_r)(\Sigma)\approx k_1(\Sigma)\times\cdots\times k_r(\Sigma)$ is a product of discrete Hodge algebras $k_i(\Sigma)$ over fields k_i, $i=1,\dots,r$. Therefore by Theorem 2.2, $S^{-1}q'$ and $S^{-1}q$ are extended from $S^{-1}A$. So, there are isometries $f_1: q_S\to\bar{q}\otimes_A S^{-1}R$ and $f_2: q'_S\to\bar{q}'\otimes_A S^{-1}R$ such that $\bar{f}_1=\mathrm{Id}$ ('$\bar{}$' always denotes 'modulo $(X_0,\dots,X_n)$'). If $\alpha_1=f_1^{-1}\circ(\bar{\varphi}\otimes_A\mathrm{Id})\circ f_2$, then $\bar{\alpha}_1=\bar{\varphi}_S$. Therefore we can find an s in S so that q'_s and q_s are extended from A_s and $\alpha_1: q'_s\xrightarrow{\sim} q_s$, an isometry with $\bar{\alpha}_1=\bar{\varphi}_s$.

Write $S'=1+sA$ and $R'=S'^{-1}A(\Sigma)$. As radical $(S'^{-1}A)$ contains a nonzero divisor s, there is a generalized dimension function $d:\operatorname{Spec} R'\to\mathbb{N}$ with $d(\mathscr{Y})<\dim R'\leq\dim R$ for all $\mathscr{Y}$ in $\operatorname{Spec} R'$ (see [1, Lemma 1.13]).

Write $q_0=(Q_0,q_0)$, then $q=(Q_0\oplus P\oplus P^*,q)$ and $(0,1)=(tz,tx,tf,1+ty)$ with (z,x,f) in $Q_0\oplus P\oplus P^*$. Since t is a nonzero divisor in $R'=S'^{-1}A(\Sigma)$, as in the proof of [4, Theorem 4.3] we can find an isometry $\alpha_2: q'_{S'}\to q_{S'}$ with $\alpha_2\equiv\varphi_{S'}$ (modulo t) and hence $\bar{\alpha}_2=\bar{\varphi}_{S'}$. So we can find an a in A such that $\alpha_2: q'_{1+as}\xrightarrow{\sim} q_{1+as}$ is defined with $\bar{\alpha}_2=\bar{\varphi}_{1+as}$.

Therefore with $s_1=s$ and $s_2=1+as$ we have $\alpha_i: q'_{s_i}\to q_{s_i}$, $i=1,2$, two isometries such that

(i) $(\alpha_1)_{s_2}\equiv(\alpha_2)_{s_1}$ modulo $(X_0,\dots,X_n)$ and

(ii) $q_{s_1s_2}$ is extended from $A_{s_1s_2}$.

Now the theorem follows from the next lemma.

Lemma 2.5. *Let $R=A(\Sigma)=A[X_0,\dots,X_n]/I(\Sigma)$ be a discrete Hodge algebra over a commutative ring A (Σ is a simplicial subcomplex of Δ_n). Let q and q' be two quadratic R-spaces and s_1,s_2 be two elements in A with $As_1+As_2=A$. Suppose $\alpha_i: q_{s_i}\to q'_{s_i}$, $i=1,2$, are two isometries with*

(1) $(\alpha_1)_{s_2}\equiv(\alpha_2)_{s_1}$ modulo $(X_0,\dots,X_n)$,

(2) $q_{s_1s_2}$ is extended from $A_{s_1s_2}$.

Then $q\approx q'$.

Proof. Note that $X_i \to X_i T$ defines a ring homomorphism $f: R \to R[T]$. Write $q=(Q,q)$, $\bar{Q}=Q/(X_0,\dots,X_n)Q$ and $\alpha=(\alpha_2)_{s_1}^{-1}(\alpha_1)_{s_2}$. As $q_{s_1s_2}$ is extended, we can consider α as an element in $\mathrm{End}(\bar{Q})\otimes R_{s_1s_2}$. Let φ be the image of α in $\mathrm{End}(\bar{Q})\otimes R_{s_1s_2}[T]$ under the map $\mathrm{Id}\otimes f$. Since $\alpha\equiv\mathrm{Id}$ modulo $(X_0,\dots,X_n)$, it follows that $\varphi(0)=\mathrm{Id}$ and $\varphi(1)=\alpha$. One can also check that φ is an isometry. Hence α is isotopic (see [4] for definition) to Id. So $(\alpha_1)_{s_2}$ is isotopic to $(\alpha_2)_{s_1}$. Now the lemma follows from [4, Lemma 4.1]. □□

Acknowledgement

I thank Parvin Sinclair for many useful discussions.

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